

Non-Linear Generalization of the Relativistic Schrödinger Equations

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Z. Naturforsch. **51 a**, 965–982 (1996); received November 11, 1995

The theory of the Relativistic Schrödinger Equations is further developed and extended to *non-linear* field equations. The technical advantage of the Relativistic Schrödinger approach is demonstrated explicitly by solving the coupled Einstein–Klein–Gordon equations including a *non-linear* Higgs potential in case of a Robertson–Walker universe. The numerical results yield the effect of dynamical self-diagonalization of the Hamiltonian which corresponds to a kind of quantum de-coherence being enabled by the inflation of the universe.

1. Introduction

Being asked what is the most successful theory in their field, most physicists would surely point out to quantum mechanics without any hesitation. But the next question, which structural element of this theory does contribute most to its overwhelming success, could presumably not be answered with equal ease. Nevertheless, there can be little doubt that the *linearity* of the traditional wave equations (Schrödinger, Dirac, Klein–Gordon etc.) would occupy a prominent position among the possible answers. And indeed, without that nice property of linearity there would be no *superposition principle* which is frequently evoked in order to account for the well-known interference effects or in order to expand some quantity with respect to a suitable basis system (linear eigenvalue problems)!

But despite this key feature of linearity within the traditional approach to quantum mechanics, it seems not possible to dispense completely with the use of non-linear wave equations. For instance, from a more historic point of view one may think here of Heisenberg's attempt to describe the truly fundamental particles by means of a *non-linear spinor equation* [1]. Or, as a more recent example, one could refer to the treatment of an open (dissipative) quantum system which must obey a *non-linear Schrödinger equation* [2]. Finally, it must be remarked that the success of the unification of the electro-weak forces is essentially based upon the use of a *non-linear Higgs field*

(“Higgs–Kibble mechanism” [3]). And not to forget: the modern inflation theories for the early evolution of the universe are mostly based upon some *non-linear scalar field* whose intermediate persistence in the “false vacuum” drives the universe's exponential growth [4].

In view of this situation it seems sufficiently interesting to inquire into the question whether the recently established theory of the “*Relativistic Schrödinger Equations*” [5, 6] can also be generalized to the *non-linear* case. Observe here that the Relativistic Schrödinger Equation (RSE) is a kind of common prototype of the traditional wave equations so that the *linear* Dirac and Klein–Gordon equations can be alternatively deduced from it. Thus, the question is now whether the original Relativistic Schrödinger Equations are capable to incorporate the *non-linear* versions of the Dirac and Klein–Gordon equations. The answer to this question is positive and the present paper is intended to demonstrate this in detail for the Klein–Gordon case (\leadsto scalar particles). On this occasion, the theory of the RSE is further elaborated and the results are applied to a concrete example: inflation of the universe by a doublet of scalar Higgs particles.

Our procedure is the following: First, we present a brief account of the present state of the RSEs as developed up to now (Section 2). Next, we extend the formalism to a gauge multiplet of N scalar particles being subject to the *non-linear* Klein–Gordon equation (Section 3).

In Sect. 4, the results obtained so far are adapted to a specific exemplifying situation, namely a Robertson–Walker universe filled with scalar Higgs matter. All dynamical equations are set up in detail and put into

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0932-0784 / 96 / 0900-0965 \$ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72072 Tübingen



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a form ready for the numerical integration. The numerical solutions (Sect. 5) exhibit some interesting effects: (i) during the inflation phase of the universe the Hubble expansion rate is large enough to cause some kind of “de-coherence” effect in the absence of the gauge forces, i. e. the Hamiltonian is subject to “dynamical self-diagonalization”. A diagonal Hamiltonian \mathcal{H}_μ is interpreted here as describing “independent” subsystems (for non-vanishing gauge forces the Hamiltonian \mathcal{H}_μ must take its value at least in the algebra of the holonomy group). (ii) during inflation the “Fierz deviation”, measuring the possibility of existence of a wave function, is violently increased. This implies that the physical densities of the Higgs system strongly deviate from that algebraic form which is due to a wave function ψ .

The paper is closed by a discussion of the technical advantages of the RSEs (Sect. 6) to which the reader is referred also for the sake of a general survey.

2. Relativistic Schrödinger Equations

Usually, relativistic quantum mechanics is based upon certain wave equations, e. g. Dirac’s equation for fermions or the Klein–Gordon equation (KGE) for scalar particles. For a single charged particle of mass M and charge e moving in an external electromagnetic field $F_{\mu\nu}$ the latter type of equation reads

$$\mathcal{D}^\mu \mathcal{D}_\mu \psi + \left(\frac{Mc}{\hbar} \right)^2 \psi = 0. \quad (2.1)$$

Here, $\psi(x)$ is a complex scalar function over space-time and \mathcal{D}_μ is the gauge-covariant derivative

$$\mathcal{D}_\mu \psi \doteq \partial_\mu \psi + \frac{ie}{\hbar c} A_\mu \psi, \quad (2.2)$$

being entered by the electromagnetic four-potential A_μ , i. e.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.3)$$

In a similar way Dirac’s equation (DE) would read

$$i\hbar c \gamma^\mu \mathcal{D}_\mu \psi = Mc^2 \psi \quad (2.4)$$

where ψ is now a 4-component spinor. Thus, matter is described by a set of rather distinct wave equations whereas the corresponding interaction fields all obey the same unified law of motion, namely the Maxwell

field equations

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c} j_\nu \quad (2.5)$$

or their non-abelian generalizations (Yang–Mills equations).

This circumstance was the motivation for searching a unified equation of motion for all the various matter fields, the RSE [5, 6]

$$i\hbar c \mathcal{D}_\mu \psi = \mathcal{H}_\mu \psi. \quad (2.6)$$

In principle, this is a 1-particle wave equation but the wave function $\psi(x)$ (bundle section) has now N components if the multiplet system under consideration consists of N intrinsic states. The generalization to other types of particles is obvious, e. g. $N = 4$ for a Dirac particle. Furthermore, the generalization to many-particle systems is performed by taking the Whitney sum of the fibre bundles for the individual particles. Correspondingly, the gauge covariant derivative \mathcal{D}_μ ,

$$\mathcal{D}_\mu \psi \doteq \partial_\mu \psi + \mathcal{A}_\mu \psi, \quad (2.7)$$

is entered now by the $N \times N$ -representation of the gauge-algebra valued potential \mathcal{A}_μ which takes account of all the possible interactions among the multiplet constituents (i. e. electromagnetic, weak, strong, etc.). With the emergence of the RSE (2.6) there arises also a new object: the Hamiltonian \mathcal{H}_μ which is a $gl(N, \mathbb{C})$ -valued 1-form and represents the crucial dynamical object within the new framework. Being a dynamical variable, the Hamiltonian \mathcal{H}_μ is itself subject to some equation of motion which, however, must be specified in such a way that two important conditions are simultaneously satisfied:

1. the Hamiltonian \mathcal{H}_μ must admit the existence of solutions $\psi(x)$ for the basic wave equation (2.6) (\leadsto “integrability condition”)
2. the Hamiltonian \mathcal{H}_μ must be determined in such a way that the validity of the basic conservation laws for the material system is guaranteed (\leadsto “conservation equation”).

Apart from these requirements, one would also like to see the new framework becoming identical to conventional quantum mechanics if applied to single-particle systems, i. e. one wants to have the KGE (2.1)

or the DE (2.4) being deduceable from the RSE (2.6). The reason is here that the predictions of the traditional wave equations (energy levels etc.) agree well with the observational data, and it is clear that this success for the 1-particle systems must not be spoiled by the new theory.

Concerning the solutions to all these problems, let us mention here briefly that the first question (integrability problem) has already found its answer in a preceding paper [6] and looks as follows: The sufficient integrability condition for the RSE (2.6) consists of the first field equation for \mathcal{H}_μ

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu + \frac{i}{\hbar c} [\mathcal{H}_\mu, \mathcal{H}_\nu] = i\hbar c \mathcal{F}_{\mu\nu} \quad (2.8)$$

to be complemented by the requirement of the commutativity of its anti-hermitean part \mathcal{L}_μ :

$$[\mathcal{L}_\mu, \mathcal{L}_\nu] = 0. \quad (2.9)$$

The (Hermitian) “localization vector field” $\mathcal{L}_\mu (= \overline{\mathcal{L}}_\mu)$ arises here by decomposition of the Hamiltonian \mathcal{H}_μ into its (anti-)Hermitian parts

$$\mathcal{H}_\mu = \hbar c (\mathcal{K}_\mu + i\mathcal{L}_\mu), \quad (2.10)$$

where the Hermitian part \mathcal{K}_μ is called the “kinetic field”. Observe also that the integrability condition for the existence of $\mathcal{H}_\mu(x)$ as a solution of (2.8) is just the Bianchi identity for the field strength $\mathcal{F}_{\mu\nu}$, which is trivially satisfied

$$\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_\mu \mathcal{F}_{\nu\lambda} + \mathcal{D}_\nu \mathcal{F}_{\lambda\mu} \equiv 0. \quad (2.11)$$

Moreover, after the existence of the solutions $\psi(x)$ has been ensured, it is even possible to specify their general shape as [6]

$$\psi(x) = \mathcal{L}(x) \cdot \mathcal{Z}^{-1}(x) \cdot \phi'. \quad (2.12)$$

This result is not so strange as it may look at a first glance: there is a constant element $\phi' \in \mathbb{C}^N$ ($\leadsto \partial_\mu \phi' \equiv 0$), and this element is acted upon by the inverse of the unitary matrix $\mathcal{Z}(x) \in \mathcal{U}(N)$. This group element is related to the gauge potential \mathcal{A}_μ (2.7) and to the kinetic field \mathcal{K}_μ (2.10) in the following way:

$$\mathcal{Z}^{-1} \cdot \partial_\mu \mathcal{Z} = \mathcal{A}_\mu + i\mathcal{K}_\mu. \quad (2.13)$$

This relation may be interpreted as the transition to a

new potential $\overset{\circ}{\mathcal{A}}_\mu$,

$$\overset{\circ}{\mathcal{A}}_\mu \doteq \mathcal{Z}^{-1} \cdot \partial_\mu \mathcal{Z}, \quad (2.14)$$

which is *trivial* in the sense that its field strength $\overset{\circ}{\mathcal{F}}_{\mu\nu}$ is identically zero

$$0 = \overset{\circ}{\mathcal{F}}_{\mu\nu} \equiv \nabla_\mu \overset{\circ}{\mathcal{A}}_\nu - \nabla_\nu \overset{\circ}{\mathcal{A}}_\mu + [\overset{\circ}{\mathcal{A}}_\mu, \overset{\circ}{\mathcal{A}}_\nu]. \quad (2.15)$$

Finally, after the constant \mathbb{C}^N -element ϕ' has been rotated by the “phase factor” \mathcal{Z}^{-1} in (2.12), the result becomes acted upon by the “modulus” $\mathcal{L}(x)$ of the wave function $\psi(x)$. This “localization scalar field” $\mathcal{L}(x)$ is a solution of the field equation

$$\overset{\circ}{\mathcal{D}}_\mu \mathcal{L} = \mathcal{L}_\mu \cdot \mathcal{L}, \quad (2.16)$$

which formally admits the singular solution (i. e. $\det \mathcal{L} = 0$)

$$\mathcal{L} = \psi \otimes \bar{\phi}; \quad \phi(x) \doteq \mathcal{Z}^{-1}(x) \cdot \phi', \quad \partial_\mu \phi' \equiv 0. \quad (2.17)$$

The reason is that the rotated \mathbb{C}^N -element $\phi(x)$ remains still covariantly constant with respect to the trivial potential $\overset{\circ}{\mathcal{A}}_\mu$ (2.14), i. e.

$$\overset{\circ}{\mathcal{D}}_\mu \phi = 0. \quad (2.18)$$

On the other hand the RSE (2.6) yields for ψ

$$\overset{\circ}{\mathcal{D}}_\mu \psi = \mathcal{L}_\mu \cdot \psi \quad (2.19)$$

when the trivial potential $\overset{\circ}{\mathcal{A}}_\mu$ (2.14) is used in place of the original \mathcal{A}_μ (2.7). Thus, differentiating the localization field $\mathcal{L}(x)$ (2.17) and using the derivatives (2.18) and (2.19) just reveals that the matrix \mathcal{L} is actually a solution of the field equation (2.16). It is especially instructive to specialize the present results to the case of a singlet scalar particle ($\leadsto N = 1$): here, the $(N \times N)$ localization matrix \mathcal{L} is simplified into a real space-time function $l(x)$, the kinetic matrix \mathcal{K}_μ becomes a simple 4-vector $k_\mu(x)$ and the $\mathcal{U}(N)$ -element $\mathcal{Z}(x)$ is a simple $\mathcal{U}(1)$ phase factor $e^{i\alpha}$, cf. (2.13):

$$e^{-i\alpha} \partial_\mu e^{i\alpha} = \frac{ie}{\hbar c} A_\mu + i k_\mu \quad (2.20a)$$

$$\leadsto \alpha(x) = \int \left(k_\mu + \frac{e}{\hbar c} A_\mu \right) dx^\mu. \quad (2.20b)$$

Since the constant number $\phi' \in \mathbb{C}^1$ may be omitted, one arrives at the usual form of a wave function in traditional quantum mechanics

$$\psi(x) = l(x)e^{-i\alpha(x)}. \quad (2.21)$$

After the existence of solutions $\psi(x)$ to the RSE (2.6) has safely been guaranteed now by the integrability condition (2.8) together with the algebraic constraint (2.9) one has to face the problem of the conservation laws to be obeyed by *any* solution $\psi(x)$. In fact there are two general conservation laws in quantum mechanics which are thought to be of such a fundamental significance that they must hold for any material system: these are the charge conservation

$$\nabla^\mu j_\mu^{(c)} \equiv 0 \quad (2.22)$$

and the energy-momentum conservation

$$\nabla^\mu T_{\mu\nu} = f_\nu. \quad (2.23)$$

The charge conservation (2.22) may be considered as a “strong” conservation law, being rigorously valid in any physical situation, whereas strict energy-momentum conservation (2.23) does hold only in the absence of external forces ($f_\nu \equiv 0$, \rightsquigarrow “weak” conservation law). In order to ensure now the validity of both fundamental conservation laws (2.22), (2.23), we have to impose a second field equation upon the Hamiltonian \mathcal{H}_μ which is expected to involve its source ($\mathcal{D}^\mu \mathcal{H}_\mu$), whereas the first field equation (2.8) referred to its curl ($\mathcal{D}_{[\mu} \mathcal{H}_{\nu]}$). In order to find the desired source equation for \mathcal{H}_μ , we first observe that both the curl and source equations are operator equations and therefore it will be convenient to first transcribe also the conservation laws (2.22), (2.23) into operator form. Thus, we introduce a velocity operator v_μ (i. e. $N \times N$ matrix), an energy-momentum operator $T_{\mu\nu}$, and a force operator f_ν . With their use we re-write the corresponding densities $j_\mu^{(c)}$ and $T_{\mu\nu}$ in the following way:

$$j_\mu^{(c)} = \bar{\psi} \cdot v_\mu \cdot \psi, \quad (2.24a)$$

$$T_{\mu\nu} = \bar{\psi} \cdot T_{\mu\nu} \cdot \psi, \quad (2.24b)$$

$$f_\nu = \bar{\psi} \cdot f_\nu \cdot \psi. \quad (2.24c)$$

From here it is a simple matter, by use of the RSE (2.6), to reformulate the conservation laws for the

densities as the corresponding condition upon the operators:

$$\mathcal{D}^\mu v_\mu + \frac{i}{\hbar c} [\bar{\mathcal{H}}_\mu \cdot v^\mu - v^\mu \cdot \mathcal{H}_\mu] = 0. \quad (2.25a)$$

$$\mathcal{D}^\mu T_{\mu\nu} + \frac{i}{\hbar c} [\bar{\mathcal{H}}^\mu \cdot T_{\mu\nu} - T_{\mu\nu} \cdot \mathcal{H}^\mu] = f_\nu. \quad (2.25b)$$

The problem is now to specify the basic operators v_μ , $T_{\mu\nu}$, and f_ν which determine with what kind of matter we are dealing. It will not come as a surprise that the choice of those operators looks rather different for Dirac particles in comparison to Klein-Gordon particles. (For a treatment of Dirac particles see [7, 8, 9].)

However, it must be stressed that the strict operator re-formulation (2.25) of the conservation laws (2.22) and (2.23) for the densities $j_\mu^{(c)}$ and $T_{\mu\nu}$ will in general be feasible only for a restricted set of solutions $\psi(x)$. The reason is that the Hamiltonian \mathcal{H}_μ strongly determines the wave function ψ entering the local densities (2.24). Therefore, the conservation laws (2.22) and (2.23) may be satisfied for the latter quantities but not for their strict operator analogues (2.25)! Thus, one has to look for an operator formulation which is completely equivalent to the corresponding density expressions. By means of some simple algebraic manipulations one is readily convinced that the desired operator version of the conservation laws must look as follows:

$$\overset{\circ}{\mathcal{D}}_\mu (\bar{\mathcal{L}} \cdot v_\mu \cdot \mathcal{L}) = 0, \quad (2.26a)$$

$$\overset{\circ}{\mathcal{D}}^\mu (\bar{\mathcal{L}} \cdot T_{\mu\nu} \cdot \mathcal{L}) = \bar{\mathcal{L}} \cdot f_\nu \cdot \mathcal{L}. \quad (2.26b)$$

Here, \mathcal{L} is the “modulus” of the wave function $\psi(x)$ (2.12) and $\overset{\circ}{\mathcal{D}}_\mu$ is the modified covariant derivative induced by the new connection $\overset{\circ}{\mathcal{A}}_\mu$ (2.14). The density relations (2.22) and (2.23) are then readily found from the operator relations (2.26) by the following link between both objects:

$$j_\mu^{(c)} = \text{tr} (\bar{\mathcal{L}} \cdot v_\mu \cdot \mathcal{L}) \equiv \text{tr} (\mathcal{I} \cdot v_\mu), \quad (2.27a)$$

$$T_{\mu\nu} = \text{tr} (\bar{\mathcal{L}} \cdot T_{\mu\nu} \cdot \mathcal{L}) \equiv \text{tr} (\mathcal{I} \cdot T_{\mu\nu}), \quad (2.27b)$$

$$f_\mu = \text{tr} (\bar{\mathcal{L}} \cdot f_\mu \cdot \mathcal{L}) \equiv \text{tr} (\mathcal{I} \cdot f_\mu), \quad (2.27c)$$

where the “intensity matrix” \mathcal{I} has been introduced by [6]

$$\mathcal{I} = \mathcal{L} \cdot \bar{\mathcal{L}}. \quad (2.28)$$

3. Klein–Gordon Particles

The preceding considerations show that for obtaining the concrete conservation laws we first have to solve certain operator equations, e. g. (2.25a) for the velocity operator v_μ in order to obtain the charge conservation. However, for this procedure one must already know the Hamiltonian \mathcal{H}_μ which itself is a dynamical object and therefore must be determined from its field equations. As the first field equation for \mathcal{H}_μ we have naturally taken the integrability condition (2.8), but this equation is not sufficient for a complete determination since only the curl of \mathcal{H}_μ is involved. Thus, we are left with the problem of finding the second field equation for \mathcal{H}_μ , which is supposed to involve its source.

Both the problems for \mathcal{H}_μ and for v_μ are now solved in one step by putting

$$v_\mu = \frac{1}{2Mc^2} (\mathcal{H}_\mu + \overline{\mathcal{H}}_\mu), \quad (3.1)$$

where M is the mass of the particle.

As a consequence, the operator equation (2.25a) for the velocity v_μ is converted into the missing equation for the Hamiltonian \mathcal{H}_μ . If the latter equation for \mathcal{H}_μ is solved for the concrete N -multiplet system under consideration, we simultaneously have also the velocity operator v_μ for the system and the charge conservation law (2.22) will hold. The particles of such a system are called “*Klein–Gordon particles*” for reasons which shall become obvious immediately.

3.1. Conservation Equation

The desired conservation equation is obtained now by substituting the Klein–Gordon ansatz (3.1) into the operator equation (2.25a) for the velocity operator, which yields the following condition:

$$\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c} \mathcal{H}^\mu \cdot \mathcal{H}_\mu = -\mathcal{D}^\mu \overline{\mathcal{H}}_\mu + \frac{i}{\hbar c} \overline{\mathcal{H}}^\mu \cdot \overline{\mathcal{H}}_\mu. \quad (3.2)$$

This equation says that the left-hand side must be anti-Hermitian, and consequently we may put

$$\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c} \mathcal{H}^\mu \cdot \mathcal{H}_\mu = -i\hbar c \mathcal{X} \quad (3.3)$$

with the Hermitian operator \mathcal{X} still to be determined. If we would specify this new object by

$$\mathcal{X} \Rightarrow \left(\frac{Mc}{\hbar} \right)^2 \cdot \mathbf{1}, \quad (3.4)$$

we would arrive at the original form for the conservation equation of the *linear* Klein–Gordon theory [5]. However, we want to leave the new operator \mathcal{X} undetermined for the moment, because charge conservation is completely independent of its special choice.

Evidently, the choice (3.1) for the velocity operator v_μ is directly related to the kinetic field \mathcal{K}_μ (2.10), i. e.

$$v_\mu \Rightarrow \left(\frac{Mc}{\hbar} \right)^{-1} \mathcal{K}_\mu. \quad (3.5)$$

This is readily seen by decomposing the Hamiltonian into its (anti-) Hermitian parts according to (2.10) and substituting this into the velocity ansatz (3.1). Thus, the conservation equation (3.3) is split up into

$$\overset{\circ}{\mathcal{D}}^\mu \mathcal{K}_\mu + \{\mathcal{K}^\mu, \mathcal{L}_\mu\} = 0 \quad (3.6a)$$

and

$$\overset{\circ}{\mathcal{D}}^\mu \mathcal{L}_\mu - i[\mathcal{K}^\mu, \mathcal{L}_\mu] + \mathcal{L}^\mu \cdot \mathcal{L}_\mu - \mathcal{K}^\mu \cdot \mathcal{K}_\mu = -\mathcal{X}. \quad (3.6b)$$

Here, the first equation (3.6a) is immediately recognized to be identical to the strict continuity requirement (2.25a), i. e.

$$\overset{\circ}{\mathcal{D}}^\mu v_\mu + \{v^\mu, \mathcal{L}_\mu\} = 0. \quad (3.7)$$

This strict form is also equivalent to the weaker version (2.26a) as long as the localization field \mathcal{L} is thought to be a regular matrix ($\det \mathcal{L} \neq 0$). Observe also that the unknown object \mathcal{X} enters only the second equation (3.6b) which is not used for charge conservation.

Summarizing results we can say that our choice of the conservation equation (3.3) as the second field equation for the Hamiltonian \mathcal{H}_μ leads to the following generalized form of the KGE:

$$\mathcal{D}^\mu \mathcal{D}_\mu \psi + \mathcal{X} \cdot \psi = 0, \quad (3.8)$$

namely by simply applying the derivative operator \mathcal{D}_μ once more to the RSE (2.6) and using just that conservation equation (3.3).

Thus, any solution $\psi(x)$ of the RSE (2.6) also obeys the (generalized) KGE (3.8), provided we additionally insist upon the validity of the strict conservation equation (3.3). However, it is immediately evident that the latter equation is not a necessary condition but only a sufficient one. Therefore, it is possible to weaken the original strict form (3.3) into

$$\left(\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c} \mathcal{H}^\mu \cdot \mathcal{H}_\mu \right) \cdot \mathcal{L} = -i\hbar c \mathcal{X} \cdot \mathcal{L}, \quad (3.3')$$

and the KGE (3.8) will still be valid! But unfortunately the equations for the Hamiltonian \mathcal{H}_μ have lost their autonomous character and now become coupled to the localization field \mathcal{L} . Consequently, one would not describe the quantum system by the “obsolete” variables of wave function ψ and gauge potential A_μ but rather in terms of the kinetic field \mathcal{K}_μ and localization field \mathcal{L} (For the coupled system of field equations for \mathcal{K}_μ and \mathcal{L} see [6]). It is even possible to further weaken the equation (3.3') into

$$\tilde{\mathcal{L}} \cdot \left(\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c} \mathcal{H}^\mu \cdot \mathcal{H}_\mu \right) \cdot \mathcal{L} = -i\hbar c \tilde{\mathcal{L}} \cdot \mathcal{X} \cdot \mathcal{L}, \quad (3.3'')$$

so that the KGE (3.8) must not necessarily be valid. But the conservation laws (2.22) and (2.23) will still hold on account of (2.26), and a wave function $\psi(x)$ will also exist as solution of the RSE (2.6). This suggests that the very concept of the RSEs is even more fundamental than the traditional wave equations (Klein–Gordon, Dirac) because in its most general form it admits to dispense ultimately with the wave function ψ and instead deals with the intensity matrix \mathcal{I} , cf. (2.27) versus (2.24). The emergence of a wave function ψ is then a special case encountered whenever the intensity matrix \mathcal{I} is of the projector type: $\mathcal{I}^2 \sim \mathcal{I}$, see [6] and the discussion of the *Fierz identity* below. But in the more general case, the RSE (2.6) for ψ would be replaced by the equation of motion for the intensity matrix \mathcal{I}

$$i\hbar c \mathcal{D}_\mu \mathcal{I} = \mathcal{H}_\mu \cdot \mathcal{I} - \mathcal{I} \cdot \overline{\mathcal{H}}_\mu. \quad (2.6')$$

3.2. Lorentz force

Next, we have to face the problem of (weak) energy–momentum conservation (2.25b). The point here is that we should be able to find an operator $T_{\mu\nu}$ such that the force operator f_ν is just of the Lorentz

type, i. e. the product of the external field strength $\mathcal{F}_{\mu\nu}$ and Hamiltonian \mathcal{H}_μ :

$$f_\nu = i \frac{\hbar}{2Mc} \left(\mathcal{F}_{\mu\nu} \cdot \mathcal{H}^\mu + \overline{\mathcal{H}}^\mu \cdot \mathcal{F}_{\mu\nu} \right). \quad (3.9)$$

The reason why we want to postulate this force is a kind of unifying point of view: according to the current belief our world is dualistic in the sense that it consists of matter (ψ) and interactions (A_μ) and therefore we need a closed dynamical system for both constituents (ψ, A_μ). As the dynamical equation for the interaction field A_μ we take the generalized Maxwell equation

$$\mathcal{D}^\mu \mathcal{F}_{\mu\nu} = \frac{4\pi}{c} \mathcal{J}_\nu. \quad (3.10)$$

Further, as the dynamical equation for the matter field ψ we take the RSE (2.6). And finally the energy–momentum content of matter ($T_{\mu\nu}$) has now to be defined in such a way that the force operator f_ν (2.25b) closes the system and does not bring in new dynamical quantities, i. e. it should be built up by the field strength $\mathcal{F}_{\mu\nu}$ and the Hamiltonian \mathcal{H}_μ alone and nothing else! This procedure naturally leads to the *Lorentz postulate* (3.9).

Let us insert here a few comments about the mathematical structure of the Lorentz force (3.9). How is it related to the gauge algebra valued current \mathcal{J}_μ occurring in the generalized Maxwell equation (3.10)? Observe that the original Lorentz force in ordinary (Maxwellian) electrodynamics is the product of the field strength and the same current that also emerges as source for the field strength. Therefore: is there a similar relationship between the generalized source \mathcal{J}_μ (3.10) and the generalized Lorentz force f_ν (3.9)?

To answer this question we observe that the curvature $\mathcal{F}_{\mu\nu}$ is a gauge–algebra valued 2–form and may therefore be decomposed with respect to the generators τ^a ($= -\overline{\tau}^a$) of the gauge algebra as follows:

$$\mathcal{F}_{\mu\nu} = \frac{q}{\hbar c} \cdot F_{a\mu\nu} \tau^a. \quad (3.11)$$

Here, q is the charge for any type of interaction specified by the generators τ^a .

Similarly to the decomposition of the curvature $\mathcal{F}_{\mu\nu}$ (3.11), there is also a decomposition of the current \mathcal{J}_μ entering the Maxwell equations (3.10), i. e.

$$\mathcal{J}_\mu = \frac{q}{\hbar c} j_{a\mu} \tau^a. \quad (3.12)$$

Thus the generalized Maxwell equations (3.10) read in component form

$$\begin{aligned} D^\mu F_{a\mu\nu} &= \frac{4\pi}{c} j_{a\nu}; \\ (D^\mu F_{a\mu\nu} &\doteq \nabla^\mu F_{a\mu\nu} + C_{ab}{}^c A^{b\mu} F_{c\mu\nu}), \\ ([\tau^a, \tau^b] &= C^{ab}{}_c \tau^c). \end{aligned} \quad (3.13)$$

Here, in the true spirit of the Maxwell–Lorentz idea, the “gauge current densities” $j_{a\mu}$ must now be related in some way to the Lorentz force operator f_ν (3.9). To this end we define the “gauge velocity operators” $v^a{}_\mu (= \overline{v}^a{}_\mu)$ through

$$v^a{}_\mu = \frac{i}{2Mc^2} (\tau^a \cdot \mathcal{H}_\mu + \overline{\mathcal{H}}_\mu \cdot \tau^a) \quad (3.14)$$

and then the decomposition (3.11) for the curvature $\mathcal{F}_{\mu\nu}$ re-casts the Lorentz force operator f_ν into the following form

$$f_\nu = q F_{a\mu\nu} v^{a\mu}. \quad (3.15)$$

(The group indices a, b, c, \dots are raised and lowered merely for aesthetic reasons.) Thus, the Lorentz force density \mathbf{f}_ν is found as the expected result

$$\mathbf{f}_\nu \doteq \overline{\psi} \cdot f_\nu \cdot \psi = \frac{1}{c} F_{a\mu\nu} j^{a\mu} \quad (3.16)$$

provided the gauge current densities $j^a{}_\mu$ are defined through

$$j^a{}_\mu = cq \overline{\psi} \cdot v^a{}_\mu \cdot \psi. \quad (3.17)$$

Clearly, we shall now identify both gauge currents $j_{a\mu}$ (3.17) and (3.12) in order to close our dynamical system. Observe, however, that these *gauge* densities (3.17) may not be considered as being equipped with a direct observable meaning in the same way as the *physical* densities, e. g. $j_\mu^{(c)}$ (2.24a) being a truly gauge *invariant* object. Remember here that in case of abelian electrodynamics the generator is $\tau^a \rightarrow \tau^{\text{el}} = -i \cdot \mathbf{1}$, and the corresponding gauge operator $v^a{}_\mu$ (3.14) reduces to our original ansatz (3.1). Therefore the corresponding current density j_μ^{el} (3.17) becomes the ordinary electromagnetic current $j_\mu^{(c)}$ which is $\mathcal{U}(1)$ gauge *invariant*, and hence j_μ^{el} may be counted as an observable quantity!

Finally, we ask whether the gauge currents $j^a{}_\mu$ (3.17) also obey some continuity equation as does the physical current $j_\mu^{(c)}$ according to (2.22), i. e. we have to inquire whether the continuity equation

$$D^\mu j^a{}_\mu = 0 \quad (3.18)$$

holds for the gauge currents. The answer is yes because the curvature $\mathcal{F}_{\mu\nu}$ obeys the identity

$$D^\mu D^\nu \mathcal{F}_{\mu\nu} \equiv 0, \quad (3.19)$$

which implies as a consequence of Maxwell’s equation (3.10)

$$D^\mu \mathcal{J}_\mu = 0. \quad (3.20)$$

If decomposed via (3.12) this is just the component version (3.18). On the other hand the gauge currents $j^a{}_\mu$ have already been defined in (3.17) with reference to the Hamiltonian \mathcal{H}_μ , and the question is now whether the conservation equation (3.3) is really consistent with that gauge conservation law (3.18)? However, some simple computations readily confirm the desired consistency via the gauge analogue of the operator equation (2.25a) for v^μ , namely

$$D^\mu v^a{}_\mu + \frac{i}{\hbar c} [\overline{\mathcal{H}}^\mu \cdot v^a{}_\mu - v^a{}_\mu \cdot \mathcal{H}^\mu] \stackrel{!}{=} 0. \quad (3.21)$$

Fortunately, this requirement can really be satisfied for our gauge velocity operators $v^a{}_\mu$ (3.14). The proof runs similarly as for the convection operator v_μ (3.1) provided the generators τ^a commute with the generalized mass operator \mathcal{X} :

$$[\tau^a, \mathcal{X}] = 0. \quad (3.22)$$

Thus, we arrive at a complete and consistent dynamical system for the N -multiplet system under consideration, and we are left with the problem of finding its energy–momentum content.

3.3. Energy–momentum density

The search for the correct energy–momentum operator $\mathcal{T}_{\mu\nu}$ will readily reveal that the *strict* operator formalism (2.25) excludes many interesting possibilities, e. g. the *non-linear* Klein–Gordon theory (3.8). On the other hand, the *weak* operator frame–work (2.26) can take into account those possibilities in a

natural way. Let us first demonstrate the shortcoming of the strict formalism.

In order to correctly produce the Lorentz force (3.9) via the *strict* operator equation (2.25b) we try the following ansatz for the energy–momentum operator $T_{\mu\nu}$ for the Klein–Gordon multiplet:

$$T_{\mu\nu} = \frac{1}{2Mc^2} (\overline{\mathcal{H}}_\mu \cdot \mathcal{H}_\nu + \overline{\mathcal{H}}_\nu \cdot \mathcal{H}_\mu) - \frac{1}{2Mc^2} g_{\mu\nu} (\overline{\mathcal{H}}^\lambda \cdot \mathcal{H}_\lambda - \mathcal{Y}), \quad (3.23)$$

where the unknown operator \mathcal{Y} is still to be determined. Now we can run through the computations as required by (2.25b) by using all the preceding results, and we will end up just with our desired Lorentz force (3.9) provided we impose upon the new element \mathcal{Y} the constraint

$$\begin{aligned} \mathcal{D}_\nu \mathcal{Y} + \frac{i}{\hbar c} [\overline{\mathcal{H}}_\nu \cdot \mathcal{Y} - \mathcal{Y} \cdot \mathcal{H}_\nu] \\ = i\hbar c [\overline{\mathcal{H}}_\nu \cdot \mathcal{X} - \mathcal{X} \cdot \mathcal{H}_\nu]. \end{aligned} \quad (3.24)$$

It is true, this operator equation for \mathcal{Y} looks very complicated, but a closer inspection of the corresponding integrability condition

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu] \mathcal{Y} \equiv [\mathcal{F}_{\mu\nu}, \mathcal{Y}] \quad (3.25)$$

readily reveals that the object \mathcal{X} must always be co-variantly constant:

$$\mathcal{D}_\mu \mathcal{X} \equiv 0. \quad (3.26)$$

Thus, we naturally put

$$\mathcal{X} = \left(\frac{Mc}{\hbar} \right)^2 \cdot \mathbf{1}, \quad (3.27)$$

and therefore the application of our Lorentz postulate (3.9) to the energy–momentum ansatz (3.23) for $T_{\mu\nu}$ inevitably leads to the *linear* Klein–Gordon theory (2.1)!

The possible solutions \mathcal{Y} for that linear case are also readily found by putting

$$\mathcal{Y} = (Mc^2)^2 \cdot \mathbf{1} + \mathring{\mathcal{Y}}, \quad (3.28)$$

where $\mathring{\mathcal{Y}}$ must now obey the equation

$$\mathcal{D}_\mu \mathring{\mathcal{Y}} + \frac{i}{\hbar c} [\overline{\mathcal{H}}_\mu \cdot \mathring{\mathcal{Y}} - \mathring{\mathcal{Y}} \cdot \mathcal{H}_\mu] = 0 \quad (3.29)$$

to be deduced from the original one (3.24). Surely, there are existing solutions $\mathring{\mathcal{Y}}$, but they are somewhat trivial, as the following argument shows: first, observe that any solution $\mathring{\mathcal{Y}}$ yields a contribution $T_{\mu\nu}^{(v)}$ to the energy–momentum density $T_{\mu\nu}$ (2.24b) of the following form, involving the Riemannian metric tensor $g_{\mu\nu}$:

$$T_{\mu\nu}^{(v)} = \frac{1}{2Mc^2} g_{\mu\nu} (\overline{\psi} \cdot \mathring{\mathcal{Y}} \cdot \psi). \quad (3.30)$$

Next, consider the derivative of the bracket density occuring here and find by means of the RSE (2.6)

$$\begin{aligned} \partial_\mu (\overline{\psi} \cdot \mathring{\mathcal{Y}} \cdot \psi) = \\ \overline{\psi} \cdot \left(\mathcal{D}_\mu \mathring{\mathcal{Y}} + \frac{i}{\hbar c} [\overline{\mathcal{H}}_\mu \cdot \mathring{\mathcal{Y}} - \mathring{\mathcal{Y}} \cdot \mathcal{H}_\mu] \right) \cdot \psi. \end{aligned} \quad (3.31)$$

Now remember the previous result (3.29) and find that this bracket density must be a constant throughout space–time:

$$\partial_\mu (\overline{\psi} \cdot \mathring{\mathcal{Y}} \cdot \psi) = 0. \quad (3.32)$$

Since the constant has the dimension of an energy density we introduce a typical length (a) and then find that the operator formalism for the Klein–Gordon particles at most admits the occurrence of a cosmological term

$$T_{\mu\nu}^{(v)} = \frac{1}{4} \frac{Mc^2}{a^3} g_{\mu\nu}, \quad (3.33)$$

but otherwise enforces a strictly linear field equation, cf. (2.1).

This negative outcome says that *non-linear* generalizations of the Klein–Gordon theory are not possible if we consider the strict conservation equation (3.3) as the basic element of relativistic quantum mechanics, together with the first field equation for \mathcal{H}_μ (2.8). However, non-linear generalizations become possible if we resort to the weak operator formalism (2.26). As a consequence of this preference of the weak version

over the strict operator formalism, the strict equation (3.24) for determining the operator \mathcal{Y} is modified into

$$\begin{aligned} \overset{\circ}{D}_\mu (\bar{\mathcal{L}} \cdot \mathcal{Y} \cdot \mathcal{L}) = \\ \hbar^2 c^2 (\bar{\mathcal{L}} \cdot \{\mathcal{L}_\mu, \mathcal{X}\} \cdot \mathcal{L} + i \bar{\mathcal{L}} \cdot [\mathcal{K}_\mu, \mathcal{X}] \cdot \mathcal{L}). \end{aligned} \quad (3.34)$$

This equation is far less restrictive than its original counterpart (3.24) which is readily seen by explicitly writing down some non-trivial solution: Putting here

$$[\mathcal{K}_\mu, \mathcal{X}] = 0 \quad (3.35)$$

and observing the gradient relationship (2.16) for \mathcal{L}_μ yields

$$\begin{aligned} \overset{\circ}{D}_\mu (\bar{\mathcal{L}} \cdot \mathcal{Y} \cdot \mathcal{L}) = \\ \hbar^2 c^2 \left(\overset{\circ}{D}_\mu (\bar{\mathcal{L}} \cdot \mathcal{X} \cdot \mathcal{L}) - \bar{\mathcal{L}} \cdot \left(\overset{\circ}{D}_\mu \mathcal{X} \right) \cdot \mathcal{L} \right), \end{aligned} \quad (3.36)$$

and this equation is solved non-trivially by means of the ansatz

$$\mathcal{Y} = Y(\rho) \cdot \mathbf{1}, \quad \mathcal{X} = X(\rho) \cdot \mathbf{1}; \quad (3.37)$$

$$\rho = \text{tr} (\bar{\mathcal{L}} \cdot \mathcal{L}),$$

i. e. the functions X and Y obey the ordinary differential equation

$$\rho \frac{dY}{d\rho} + Y = (\hbar c)^2 X. \quad (3.38)$$

A special solution hereof would be the linear Klein-Gordon theory, where X and Y are independent of the scalar density ρ :

$$X \Rightarrow \left(\frac{Mc}{\hbar} \right)^2; \quad Y \Rightarrow (Mc^2)^2. \quad (3.39)$$

However, it is possible to substitute for X any reasonable *non-linear* function of the density ρ , and for the sake of physical relevance we exemplify this by resorting to a certain non-linear potential underlying the original inflation theory of the universe [4]

$$X(\rho) = 2 \left(\frac{Mc}{\hbar} \right)^2 (2a^3 \rho - 1), \quad (3.40a)$$

$$Y(\rho) = 2 (Mc^2)^2 (a^3 \rho - 1) \quad (3.40b)$$

with a constant length parameter a . Thus, the non-linear contribution $T_{\mu\nu}^{(Y)}$ to the energy-momentum density $T_{\mu\nu}$ is read off from (3.23) and (3.49b) as

$$\begin{aligned} T_{\mu\nu}^{(Y)} &= \frac{1}{2Mc^2} g_{\mu\nu} (\bar{\psi} \cdot \mathcal{Y} \cdot \psi) \\ &\Rightarrow g_{\mu\nu} Mc^2 \rho (a^3 \rho - 1). \end{aligned} \quad (3.41)$$

Adding the vacuum term $T_{\mu\nu}^{(v)}$ (3.33) we end up with the result

$$\begin{aligned} T_{\mu\nu}^{(Y)} + T_{\mu\nu}^{(v)} &= g_{\mu\nu} Mc^2 a^3 \left(|\psi|^2 - \frac{1}{2a^3} \right)^2 \\ &\equiv g_{\mu\nu} V_X(\rho), \\ &\text{where } |\psi| \doteq \sqrt{\rho}. \end{aligned} \quad (3.42)$$

The non-linear potential $V_X(\rho)$ occurring here has the well-known shape (\sim ‘‘Higgs-potential’’)

$$V_X(\rho) = Mc^2 a^3 \left(\rho - \frac{1}{2a^3} \right)^2 \quad (3.43)$$

and was used in this form in the foundation of inflation theory [4]. We shall use this potential now in order to study the effects occurring during the inflatory phase of the universe’s evolution.

4. Expanding universe

As a demonstration of the usefulness of the present approach we consider now a Robertson-Walker universe whose energy-momentum content $T_{\mu\nu}$ is exclusively due to the non-linear Klein-Gordon field ψ treated in the preceding section. For such a physical situation the theory of the RSEs can develop its full potentialities. The reason is that the energy-momentum density $T_{\mu\nu}$ emerging on the right-hand side of Einstein’s equation

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 8\pi \frac{L_p^2}{\hbar c} T_{\mu\nu} \\ (L_p \dots \text{Planck length}) \end{aligned} \quad (4.1)$$

contains some of the physical densities due to the Klein-Gordon field ψ . But the equations of motion for these densities are just governed by the Hamiltonian \mathcal{H}_μ . In this way, one readily arrives at the coupled Einstein-Klein-Gordon system of equations which necessarily is a *first-order system* with respect to the densities of the Klein-Gordon field ψ . This system

can be integrated numerically and the results show the typical effects of inflation theory: phase transition and energy-production.

4.1. Robertson–Walker geometry

More concretely, for our subsequent demonstration we choose a Higgs doublet

$$\psi(x) = \begin{pmatrix} \psi_I(x) \\ \psi_{II}(x) \end{pmatrix}, \quad (4.2)$$

and consequently the Hamiltonian \mathcal{H}_μ must be a $gl(2, \mathbb{C})$ -valued 1-form. But due to the high symmetry of a Robertson–Walker universe the general form of \mathcal{H}_μ can be essentially simplified. First, observe here that the Robertson–Walker symmetry enforces an energy-momentum density $T_{\mu\nu}$ of the following kind:

$$T_{\mu\nu} = W b_\mu b_\nu - P B_{\mu\nu}. \quad (4.3)$$

Here, the energy density W and pressure P are depending exclusively upon the cosmic time θ but are constant over the time slices ($\theta = \text{const}$). Moreover, the Hubble flow b_μ is just the gradient field of the cosmic time θ , i. e.

$$b_\mu = \partial_\mu \theta \quad (b^\mu b_\mu = +1), \quad (4.4)$$

and $B_{\mu\nu}$ is the orthogonal projector

$$B_{\mu\nu} b^\nu = 0, \quad (4.5a)$$

$$B_\mu{}^\nu B_{\nu\lambda} = B_{\mu\lambda}, \quad (4.5b)$$

$$B^\mu{}_\mu = 3. \quad (4.5c)$$

Thus, the homogeneity of this physical situation may be expressed through

$$\nabla_\mu b_\nu = H B_{\mu\nu}, \quad (4.6a)$$

$$B_\mu{}^\nu \partial_\nu W = B_\mu{}^\nu \partial_\nu P \equiv 0, \quad (4.6b)$$

where $H (\equiv \dot{\mathcal{R}}/\mathcal{R})$ is the Hubble expansion rate. As a consequence of this highly symmetric geometry there is left only a single dynamical object, namely the universe's radius \mathcal{R} . Its equation of motion is found from the Einstein equations (4.1) as ($\dot{\mathcal{R}} \equiv d\mathcal{R}/d\theta$)

$$\frac{\dot{\mathcal{R}}}{\mathcal{R}} = -4\pi \frac{L_p^2}{\hbar c} \left(P + \frac{1}{3} W \right), \quad (4.7a)$$

$$H^2 \equiv \left(\frac{\dot{\mathcal{R}}}{\mathcal{R}} \right)^2 = \frac{\sigma}{\mathcal{R}^2} + \frac{8\pi}{3} \frac{L_p^2}{\hbar c} W \quad (4.7b)$$

and may be supplemented by the “work–energy theorem” ($\nabla^\mu T_{\mu\nu} \equiv 0$)

$$\frac{d}{d\theta} (W \mathcal{R}^3) = -P \frac{d\mathcal{R}^3}{d\theta}. \quad (4.8)$$

Obviously, (4.7b) (“initial value equation”) specifies here a first integral of the proper equation of motion (4.7a) for the radius \mathcal{R} , see [10].

4.2. Energy and Pressure

The next problem is now to find the energy density W and pressure P in terms of the 2-particle wave function ψ (4.2) and the Hamiltonian \mathcal{H}_μ . But this is readily attained by simply observing the requirement of homogeneity and isotropy, as expressed through (4.6). Consequently, our nearby guess for the present cosmological situation is

$$\mathcal{H}_\mu = \mathcal{H} b_\mu, \quad (4.9)$$

where $\mathcal{H} = \mathcal{H}(\theta)$ is a $gl(2, \mathbb{C})$ -valued function of cosmic time θ . Clearly, such a simple form of the Hamiltonian \mathcal{H}_μ will produce also an analogously simple result for the energy-momentum operator $\mathcal{T}_{\mu\nu}$ (3.23):

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= \frac{1}{2Mc^2} (\overline{\mathcal{H}} \cdot \mathcal{H} + \mathcal{Y}) b_\mu b_\nu \\ &\quad - \frac{1}{2Mc^2} (\overline{\mathcal{H}} \cdot \mathcal{H} - \mathcal{Y}) B_{\mu\nu}. \end{aligned} \quad (4.10)$$

From this result we first deduce the “energy operator” \mathcal{W} as

$$\mathcal{W} = \frac{1}{2Mc^2} (\overline{\mathcal{H}} \cdot \mathcal{H} + \mathcal{Y}) \quad (4.11)$$

and the “pressure operator” \mathcal{P} as

$$\mathcal{P} = \frac{1}{2Mc^2} (\overline{\mathcal{H}} \cdot \mathcal{H} - \mathcal{Y}). \quad (4.12)$$

Then the energy density W is readily found as

$$\begin{aligned} W &= \overline{\psi} \cdot \mathcal{W} \cdot \psi \\ &= \frac{1}{2Mc^2} (\overline{\psi} \cdot \overline{\mathcal{H}} \cdot \mathcal{H} \cdot \psi) + \frac{1}{2Mc^2} (\overline{\psi} \cdot \mathcal{Y} \cdot \psi), \end{aligned} \quad (4.13)$$

and similarly the pressure P as

$$\begin{aligned} P &= \bar{\psi} \cdot \mathcal{P} \cdot \psi \\ &= \frac{1}{2Mc^2} (\bar{\psi} \cdot \bar{\mathcal{H}} \cdot \mathcal{H} \cdot \psi) - \frac{1}{2Mc^2} (\bar{\psi} \cdot \mathcal{Y} \cdot \psi). \end{aligned} \quad (4.14)$$

But the second term on the right hand sides has already been computed by (3.41)–(3.42), i. e.

$$\frac{1}{2Mc^2} \bar{\psi} \cdot \mathcal{Y} \cdot \psi = V_X(\rho) \quad (4.15)$$

with the Higgs potential V_X being specified in (3.43). Thus, we are left with computing the first term $(\bar{\psi} \cdot \bar{\mathcal{H}} \cdot \mathcal{H} \cdot \psi)$ in terms of the Hamiltonian \mathcal{H}_μ (2.10) and the densities

$$\rho = \bar{\psi} \cdot \psi, \quad s_j = \bar{\psi} \cdot \sigma_j \cdot \psi. \quad (4.16)$$

To this end, the reduced Hamiltonian \mathcal{H} (4.9) is split into its (anti-) Hermitian parts \mathcal{K} and \mathcal{L} in close analogy to the original decomposition (2.10), i. e. we put

$$\mathcal{H} = \hbar c (\mathcal{K} + i \mathcal{L}), \quad (4.17)$$

where

$$\mathcal{K} = \mathbf{K} \cdot \mathbf{1} + \mathbf{K}_j \sigma_j, \quad (4.18a)$$

$$\mathcal{L} = \mathbf{L} \cdot \mathbf{1} + \mathbf{L}_j \sigma_j \quad (4.18b)$$

($\sigma_j \dots$ Pauli matrices).

With these arrangements the desired bracket term $(\bar{\psi} \cdot \bar{\mathcal{H}} \cdot \mathcal{H} \cdot \psi)$ is of the form

$$\bar{\psi} \cdot \bar{\mathcal{H}} \cdot \mathcal{H} \cdot \psi = (\hbar c)^2 (h\rho + 2h_j s^j), \quad (4.19)$$

where the Hamiltonian coefficients h, h_j are given by

$$h = \mathbf{K}^2 + \mathbf{K}_j \mathbf{K}^j + \mathbf{L}^2 + \mathbf{L}_j \mathbf{L}^j, \quad (4.20a)$$

$$h_j = \mathbf{K} \mathbf{K}_j + \mathbf{L} \mathbf{L}_j - \epsilon^{kl}{}_j \mathbf{K}_k \mathbf{L}_l. \quad (4.20b)$$

Therefore, the energy density W (4.13) and the pressure P (4.14) are finally found in terms of the densities (ρ, s^j) and the Hamiltonian coefficients (h, h_j) as

$$W = \frac{\hbar^2}{2M} (h\rho + 2h_j s^j) + V_X(\rho), \quad (4.21a)$$

$$P = \frac{\hbar^2}{2M} (h\rho + 2h_j s^j) - V_X(\rho). \quad (4.21b)$$

If this result for W and P is substituted into the Einstein equations (4.7) one obtains the dynamical equations for the radius \mathcal{R} of the universe which however must be complemented by the corresponding dynamical equations for the densities (ρ, s_j) and for the Hamiltonian coefficients (h, h_j) in order to close the total dynamical system.

4.3. Physical densities

Turning first to the densities let us remark that the RSE (2.6) always establishes a coupled first-order system of equations among them. This first-order system can be used in order to determine the wave function ψ itself [9]. But in the present context we do not explicitly need the wave function ψ , and therefore we are satisfied with the dynamics of the densities $\{\rho, s_j\}$. The reason is that in Einstein's theory of gravity (4.1) the space-time geometry reacts only to the densities via energy W and pressure P (4.3) but not to the wave function ψ directly. Nevertheless, one can exploit the fact that the densities are generated by some wave function ψ , cf. (4.16), in order to deduce the field equations for the densities, e. g. for the scalar density ρ :

$$\begin{aligned} \partial_\mu \rho &= \partial_\mu (\bar{\psi} \psi) = (\mathcal{D}_\mu \bar{\psi}) \cdot \psi + \bar{\psi} \cdot (\mathcal{D}_\mu \psi) \\ &= \frac{i}{\hbar c} \bar{\psi} [\bar{\mathcal{H}}_\mu - \mathcal{H}_\mu] \psi = 2\bar{\psi} \cdot \mathcal{L}_\mu \cdot \psi. \end{aligned} \quad (4.22)$$

Observing here the arrangements mentioned above readily yields

$$\begin{aligned} b^\mu \partial_\mu \rho &\equiv \dot{\rho} = 2\bar{\psi} (b^\mu \mathcal{L}_\mu) \psi \\ &= 2\bar{\psi} \cdot \mathcal{L} \cdot \psi = 2\rho \mathbf{L} + 2s^j \mathbf{L}_j. \end{aligned} \quad (4.23)$$

In a similar way the dynamical equations for the remaining three densities are found as

$$\hat{\mathcal{D}}_\mu s_j = 2 (s_j \mathbf{L} + \rho \mathbf{L}_j) b_\mu. \quad (4.24)$$

Here, the gauge-covariant derivative $\hat{\mathcal{D}}$ refers to the $su(2)$ restriction $\hat{\mathcal{A}}_\mu$ of the trivial connection $\mathring{\mathcal{A}}_\mu$ (2.14), i. e.

$$\hat{\mathcal{A}}_\mu = \mathring{\mathcal{A}}_\mu \Big|_{su(2)} = -\frac{i}{2} \hat{\mathcal{A}}_{j\mu} \sigma^j, \quad (4.25a)$$

$$\hat{\mathcal{D}}_\mu s_j = \partial_\mu s_j + \epsilon_j{}^{kl} \hat{\mathcal{A}}_{k\mu} s_l. \quad (4.25b)$$

However, some caution is necessary here because the proper dynamical equation for s_j must refer to the $su(2)$ restriction $\hat{\mathcal{A}}_\mu = \mathcal{A}_\mu|_{su(2)}$ which yields

$$b^\mu \hat{\mathcal{D}}_\mu s_j \equiv \dot{s}_j = 2 (s_j \mathcal{L} + \rho \mathcal{L}_j - \epsilon_j^{kl} s_l \mathcal{K}_k). \quad (4.26)$$

Observe also that the field equations (4.23, 4.24) of the densities are entered exclusively by the localization field \mathcal{L} but not by the kinetic field \mathcal{K} provided all the gauge-covariant derivatives are referred to the appropriate connection ($\hat{\mathcal{A}}_\mu$).

There is also a further interesting point in connection with the densities: the question of the “Fierz identities” [6, 9]. It can easily be shown that whenever the densities are generated by means of a wave function ψ , such as e. g. in (4.16), then they must obey a certain set of identities. For the present case of a two-component wave function (4.2) it is easy to check that the corresponding identity must read

$$\rho^2 - s^j s_j \equiv 0. \quad (4.27)$$

However, the theory of the RSEs admits a certain generalization, so that one can dispense with the existence of a wave function ψ (see the arguments leading to (2.6')) and therefore the “Fierz identity” (4.27) must not hold in the generalized situation. Nevertheless, the field equations (4.23), (4.26) for the densities are still valid in the more general context. Hence the Fierz identity (4.27) selects a special subset of solutions (ρ, s_j) to the field equations (4.23), (4.26), namely just those solutions admitting the reconstruction of the original wave function ψ (4.2). (For the reconstruction of the four-component Dirac wave function from the corresponding densities see [11].) But in the quite general case one finds here by use of the field equations (4.23, 4.26)

$$\frac{d}{d\theta} (\rho^2 - s^j s_j) = 4\mathcal{L} (\rho^2 - s^j s_j), \quad (4.28)$$

which is readily integrated to yield the “Fierz deviation” Δ_F^2

$$\Delta_F^2 \doteq \rho^2 - s^j s_j = (\rho_*^2 - s_{j*} s_{j*}) \cdot \exp \left[4 \int_{\theta_*}^{\theta} \mathcal{L}(\theta') d\theta' \right] \\ (\rho_*, s_{j*} = \text{const}). \quad (4.29)$$

Therefore, whenever the densities obey the Fierz iden-

tity (4.27) at the initial time θ_* , then the Fierz identity is satisfied for all time θ (provided \mathcal{L} remains non-singular). But when the Fierz identity is not obeyed initially there arises the question whether its invalidation becomes worse or whether the Fierz identity is approximated asymptotically in the course of increasing cosmic time θ ?

4.4. Hamiltonian dynamics

Before we study the answer to this question in the next section let us first close here our dynamical system, which up to now merely consists of the Einstein equations (4.7) for the universe’s radius \mathcal{R} and of the field equations (4.23), (4.26) for the densities $\{\rho, s_j\}$. However, the latter equations contain also the Hamiltonian (localization) coefficients $\{\mathcal{L}, \mathcal{L}_j\}$, and the thermodynamic state functions P and W (4.21) additionally contain the Hamiltonian (kinetic) coefficients \mathcal{K} and \mathcal{K}_j . Consequently, we have to exploit the Hamiltonian dynamics (2.8) and (3.3) in order to find the equations of motion for the remaining variables $\{\mathcal{K}, \mathcal{K}_j, \mathcal{L}, \mathcal{L}_j\}$, cf. (4.18).

First consider the integrability condition (2.8) which requires the specification of the nature of the gauge force $\mathcal{F}_{\mu\nu}$. Since the wave function ψ of our physical system has just two components, cf. (4.2), it would be capable of feeling the electro-weak force where the field strength $\mathcal{F}_{\mu\nu}$ takes its values in the corresponding gauge algebra $u(1) \oplus su(2)$. However, the existence of a non-vanishing electromagnetic (i. e. Maxwellian) $u(1)$ field $F_{\mu\nu}$ would break the Robertson-Walker symmetry and therefore we have to omit it here. Furthermore, the presence of a non-vanishing weak $su(2)$ field ($\mathcal{F}_{\mu\nu} \Rightarrow -\frac{i}{2} F_{j\mu\nu} \sigma^j$) would spoil our symmetrical ansatz (4.9) because the integrability condition (2.8) says that the Hamiltonian \mathcal{H}_μ can never take its values in some subspace which is smaller than the gauge algebra (here $su(2)$). Therefore, we want to omit here completely any gauge force for the sake of simplicity. Observe also that the coupling to gravity occurs along the Einstein equations (4.1) and not along the principles of the gauge interactions. It is well-known that the consistency of the Einstein equations requires the energy-momentum density $T_{\mu\nu}$ to be source-free (\leadsto vanishing Lorentz force (3.9)) and exactly this is ensured here by our choice of a vanishing field strength $\mathcal{F}_{\mu\nu}$! As a consequence the integrability condition (2.8) is obeyed trivially by our special ansatz (4.9), and this implies that the desired

equations of motion for the Hamiltonian coefficients K , K_j , L , L_j can be deduced exclusively from the conservation equation (3.3).

Introducing the decompositions (4.18) of our Hamiltonian \mathcal{H}_μ (4.9) into that equation (3.3) and observing the derivative of the Hubble flow b_μ (4.6a) readily yields the desired equations of motion as

$$\dot{\mathcal{H}} + 3H\mathcal{H} - \frac{i}{\hbar c} \mathcal{H}^2 = -i\hbar c \mathcal{X} \left(\dot{\mathcal{H}} \doteq \frac{d\mathcal{H}}{d\theta} \right), \quad (4.30)$$

or in components:

$$\dot{K} + 3HK = -2(LK + L_j K^j), \quad (4.31a)$$

$$\dot{L} + 3HL = -X(\rho) + K^2 - L^2 + K_j K^j - L_j L^j, \quad (4.31b)$$

$$\dot{K}_j + 3HK_j = -2(KL_j + LK_j), \quad (4.31c)$$

$$\dot{L}_j + 3HL_j = 2(KK_j - LL_j). \quad (4.31d)$$

From here it may be readily seen that the diagonal configuration for \mathcal{H}_μ ($\leadsto K_1 = K_2 = L_1 = L_2 = 0$) is always a special solution of the problem but it is not the general solution! Imagine that our doublet system starts in a truly “coherent” (i. e. non-diagonal) state $K_j \neq 0$, $L_j \neq 0$: what is its further evolution? Will the “coherence” be maintained or will it decay into some state of “independence” (i. e. diagonal Hamiltonian \mathcal{H}_μ)? For the latter situation there are obviously two possible routes for the system in order to arrive at this final state of “independence”: (i) both the kinetic field K_j and the localization field L_j become (anti-) parallel ($K_j \sim L_j$) such that the Hamiltonian \mathcal{H} (4.17) becomes diagonalizable or (ii) both fields K_j and L_j simultaneously tend to zero ($K_j K^j \rightarrow 0$, $L_j L^j \rightarrow 0$) such that the Hamiltonian \mathcal{H} becomes proportional to unity ($\mathcal{H} \sim 1$). For any of the two routes the Higgs doublet ψ (4.2) becomes a composite of *dynamically disentangled constituents* ψ_1 and ψ_2 .

The numerical solutions to (4.31) must now decide which route is taken.

5. Numerical results

Before going into the numerical details let us first have a look upon some quite general features of our model universe. According to the general belief in modern cosmology [12] the real universe started with a very small radius \mathcal{R} and high energy density W but negative pressure P . The reason for these assumptions

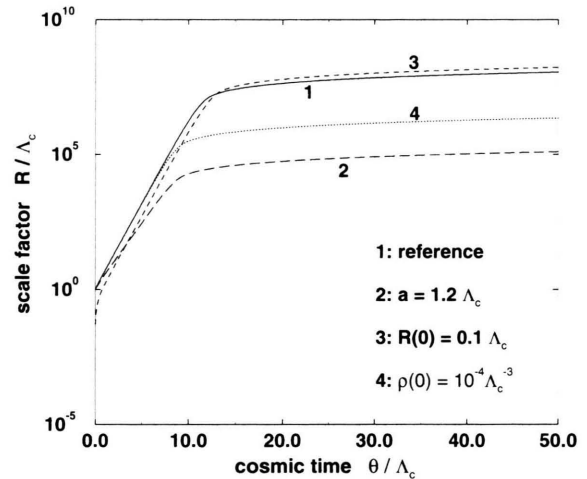


Fig. 1. *Inflation and standard phase.* During the first phase (\leadsto “inflation”) the radius \mathcal{R} grows according to the exponential law (5.2) because the false vacuum (5.1) acts as a positive cosmological constant ($\theta/\Lambda_c \lesssim 12$). When the matter field ψ evolves into the true vacuum, i. e. minimal value of the Higgs potential $V_X(\rho)$ (3.43), then there is a transition into the “standard phase” with moderate expansion rates $H = \dot{\mathcal{R}}/\mathcal{R}$ ($\theta/\Lambda_c \gtrsim 12$). The reference plot (solid line) has parameters: Planck length (L_P) = Compton length (Λ_c) = length parameter (a). The initial values are $\mathcal{R}(0) = \Lambda_c$, $\rho(0) = 10^{-5} \Lambda_c^{-3}$, $s_1(0) = 10^{-5} \Lambda_c^{-3}$, $s_2(0) = s_3(0) = 0$, $K(0) = -\Lambda_c^{-1}$, $L(0) = 2\Lambda_c^{-1}$, $L_1(0) = 0$, $L_2(0) = -L_3(0) = \Lambda_c^{-1}$, $K_1(0) = K_3(0) = 0$, $K_2(0) = \Lambda_c^{-1}$. In order to demonstrate the stability of the phase transition the dashed and dotted plots refer each time to one single variation within the set of parameters and initial values, all others being left unchanged.

is that the negative value of pressure P was capable of blowing up the tiny universe according to the Einsteinian equations (4.7), and thus the universe got its primeval outward push whose subsequent deceleration is lasting up to our present epoch (Figure 1).

5.1. Two-phase evolution

The transition from the inflation phase $P < 0$ into the standard phase ($P \gtrsim 0$) is thought to occur after the density ρ has evolved from its original small value ($\rho \approx 0$) into the equilibrium value ρ_e ($= 1/(2a^3)$) of the Higgs potential V_X (3.43). Thus, there was a very high energy density at the very beginning due to the initial value of the Higgs potential, i. e.

$$V_X(0) \sim \frac{1}{4} \frac{Mc^2}{a^3}. \quad (5.1)$$

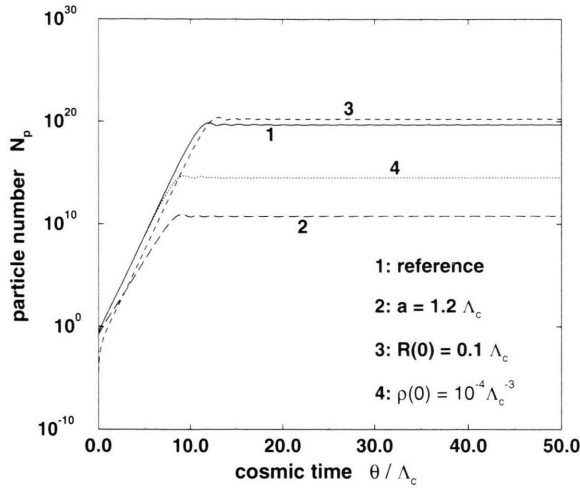


Fig. 2. *Energy production by negative pressure.* The energy $W\mathcal{R}^3$ in any co-moving 3-cell of linear size \mathcal{R} is measured by the “particle number” $N_p = W\mathcal{R}^3/Mc^2$. This number is violently raised by the negative pressure P occurring during the *inflation phase* ($\theta \lesssim 12\Lambda_c$). But in the subsequent *standard phase* ($\theta \gtrsim 12\Lambda_c$) N_p remains approximately constant at its high level because of the relative ineffectiveness of pressure P .

Here, a plausible choice for the length parameter (a) would refer to the order of magnitude of the Compton length (\hbar/Mc) corresponding to the typical mass M of that hypothetical Higgs particle. It is true that the potential energy V_X dropped down to zero from its initially high value (5.1) during the evolution of the density ρ from zero to the equilibrium value ρ_e , but this does not imply that the energy content of a co-moving 3-volume must also drop down to zero. On the contrary: the numerical integrations of the “work–energy theorem” (4.8), as part of the dynamical system, demonstrate very clearly that the “particle number” N_p ($\doteq \mathcal{R}^3 W/Mc^2$) rapidly develops up to tremendous numbers (see Figure 2). Clearly, this is the effect of the negative pressure P (4.21b) acting during the inflatory phase. But when the pressure adopts (at least on the average) its positive or zero value due to the subsequent standard phase the energy production stops, and the particle number remains at its high level attained during the inflatory phase. Simultaneously, the expansion rate H adopts more moderate positive values characteristic for the weak deceleration during the standard epoch ($\theta \gtrsim 12\Lambda_c$).

Thus the coupling of gravity to the non-linear Klein–Gordon field ψ (3.8), according to Einstein’s

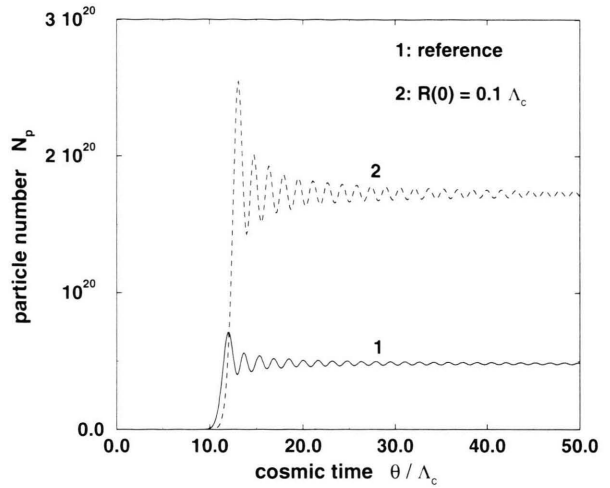


Fig. 3. *Oscillations of particle number N_p .* After the transition into the standard phase ($\theta \gtrsim 12\Lambda_c$) the particle number N_p is subject to damped oscillations of period $\sim \Lambda_c$ and relaxation time $\sim H^{-1}$. Thus, the oscillations become undamped in the late standard phase ($\theta \gtrsim 40\Lambda_c$) where the expansion rate H tends to zero.

equation (4.1), predicts two distinct phases for the evolution of the universe: During the first phase (\sim “inflation”) the radius \mathcal{R} grows exponentially, i. e.

$$\mathcal{R}(\theta) \sim \exp\left(\frac{\theta}{\theta_1}\right). \quad (5.2)$$

Here, the inflatory growth rate θ_1 can be readily determined from the Friedmann equation (4.7b) as

$$H_I^2 \equiv \frac{1}{\theta_1^2} = \frac{2\pi}{3} \frac{L_p^2}{\Lambda_c a^3}, \quad (5.3)$$

where the false vacuum value $V_X(0)$ (5.1) has been substituted for the exact energy density W (4.21a). Clearly, the exponential growth law (5.2) can be exact only for the flat universe ($\sigma = 0$) but is approximative for the open ($\sigma = +1$) and closed ($\sigma = -1$) universes. However, the inflation cannot last forever because the density ρ develops more or less slowly into its equilibrium value ρ_e ($\leadsto V_X(\rho_e) = 0$) which initiates the transition into the standard phase. However, as the oscillations of the particle number N_p of Fig. 3 show, the equilibrium value ρ_e is approximated very slowly and in an oscillatory manner (Figure 4).

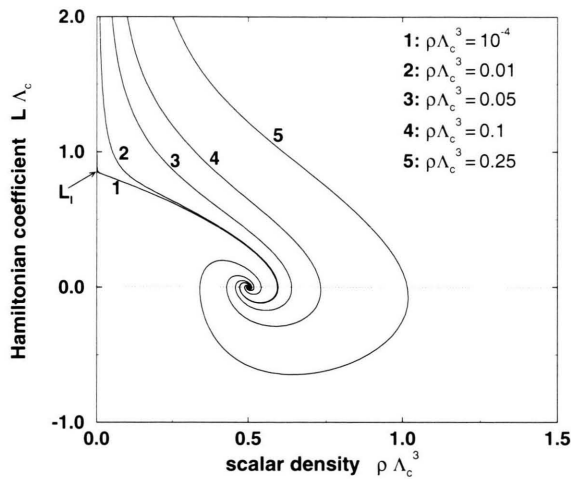


Fig. 4. *Oscillations around the true vacuum.* The subdynamics (5.4) leads to damped oscillations in the standard phase with period T_s (5.7) on the Compton scale and damping time T_D (5.6) on the Hubble scale. For continued expansion in the standard phase (“deceleration”) the Hubble rate H tends to zero and the true vacuum is asymptotically reached ($T_D \rightarrow \infty$).

5.2. Cosmic Oscillations

The origin of these oscillations lies in the subdynamics of the two variables ρ and L . The corresponding dynamical equations (4.23) for ρ and (4.31b) for L would establish a closed subsystem if the term $s^j L_j$ on the right hand side of (4.23) and the terms K^2 , $K_j K^j$, $L_j L^j$ on the right hand side of (4.31b) could be neglected. The correctness of this presumption will be readily justified below, and thus the truncated form of the oscillation subdynamics emerges in its linearized form as

$$\dot{\rho} \approx 2\rho L, \quad (5.4a)$$

$$\dot{L} + 3HL \approx -\frac{a^3}{\Lambda_c^2} (\rho - \rho_c), \quad (5.4b)$$

where the function $X(\rho)$ has been substituted from our previous choice (3.40a). Furthermore, the oscillations occur in the immediate vicinity of the true vacuum ($\rho \approx \rho_c$) and on a time scale much shorter than the Hubble time H^{-1} so that the expansion rate H may be treated as a constant ($H \rightarrow H_s = \text{const}$). By these approximations the damped oscillations are readily deduced from (5.4) as ($a \equiv \Lambda_c$)

$$\rho, L \sim \exp\left(-\frac{\theta}{T_D}\right) \cdot \sin\left(2\pi \frac{\theta}{T_s} + \delta\right) \quad (5.5)$$

with the damping time T_D being essentially given by the Hubble time H^{-1} ,

$$T_D = \frac{2}{3} H_s^{-1}, \quad (5.6)$$

and the period T_s of the oscillations being determined through

$$T_s = \frac{\Lambda_c}{\sqrt{1 - \left(\frac{3\Lambda_c H_s}{4}\right)^2}} \gtrsim \Lambda_c, \quad (5.7)$$

in good agreement with Figure 3.

5.3. De-coherence by inflation

Usually, this two-phase picture of the early universe is completed by the assumption that the oscillatory transition from the inflation into the standard phase is accompanied by some transmutation of the Higgs field energy W (4.21a) into the energy of ordinary particles forming then the hot big bang plasma. Thus, the globally coherent quantum state ψ of the Higgs field would decay into a thermodynamic ensemble of particles whose individual wave functions would no longer be correlated. However, in the literature there is no general agreement about this kind of energy transmutation [13]. In the present context of the RSEs there arises the question of “de-coherence” in a somewhat different form, namely as the problem of dynamical self-diagonalization of the Hamiltonian \mathcal{H}_μ . Observe here that for a *diagonal* Hamiltonian \mathcal{H}_μ the RSE (2.6) does no longer couple the individual components of the wave function ψ . Consequently, these “de-coherent” parts of ψ develop “independently” from each other. In this sense we now ask whether our (reduced) Hamiltonian \mathcal{H} (4.9) becomes diagonalized during the inflatory phase, either by parallelization of its kinetic and localization parts K_j , L_j (4.18) or by their simultaneous decay to zero?

The numerical integrations of the Hamiltonian system (4.31) clearly show that the Hamiltonian coefficients K , L , K_j , L_j fall apart into two classes of quite different time behaviour (Figure 5): it is true, all these coefficients very rapidly tend to zero at the first moment (up to $\theta \gtrsim \Lambda_c$). But then the localization parameter L stops decreasing and is (meta-)stabilized at a

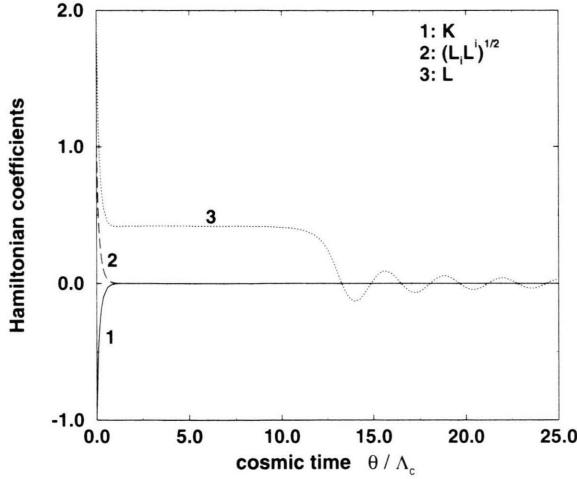


Fig. 5. *De-coherence by inflation.* During the inflation phase ($\Lambda_c \lesssim \theta \lesssim 12\Lambda_c$) the Hamiltonian coefficient L adopts its constant value L_1 (5.11) which together with the constant expansion rate H_1 (5.10) sets the time scale $T_I \approx \frac{1}{5}\Lambda_c$ for the exponential decay of the other Hamiltonian coefficients K, K_j, L_j ($\sim \exp[-\theta/T_I]$). This exponential de-coherence process comes to an end through the transition into the standard phase ($\theta \approx 12\Lambda_c$). (Parameters and initial conditions are the same as for the reference curve of Figure 1. The plot of $(K^j K_j)^{1/2}$ has been omitted because it is very similar to K .)

constant value L_1 whereas the other coefficients K, K_j, L_j continue their rapid decay to zero. However, after a sufficiently long time ($\theta \gtrsim 12\Lambda_c$), the localization variable L also leaves its constant value L_1 and approaches zero in the form of damped oscillations (Figs. 4 and 5). Comparing this time behaviour of L to that of the scale factor \mathcal{R} (Fig. 1) one readily sees that the duration of constancy of L ($\Lambda_c \lesssim \theta \lesssim 12\Lambda_c$) just agrees with the duration of the inflatory phase. Thus, the transition into the standard phase is signaled by the decay of the constant inflation value for L . In other words: the latter variable L lives roughly ten times longer than the remaining Hamiltonian coefficients L_j, K, K_j , and this gives rise to a clear distinction of the two phases, also with respect to the Hamiltonian \mathcal{H}_μ .

This circumstance may be exploited now in order to approximately compute the time behaviour of those short-living variables K, K_j, L_j during inflation: neglect the products of the short-living variables in their equations of motion (4.31), take the constant inflation value L_1 for L and then end up with the following truncated equations of motion:

$$\dot{K} = -(2L_1 + 3H_1)K, \quad (5.8a)$$

$$\dot{K}_j = -(2L_1 + 3H_1)K_j, \quad (5.8b)$$

$$\dot{L}_j = -(2L_1 + 3H_1)L_j. \quad (5.8c)$$

Since both the expansion rate H of the universe and the localization parameter L adopt constant positive values (H_1, L_1) during the inflation phase, the truncated equations (5.8) predict an exponential decay of the short-living variables for this epoch. Their decay “time” $T_I \equiv (2L_1 + 3H_1)^{-1}$ may be estimated (i) from the equation of motion (4.31b) for L yielding just for the inflational epoch

$$3H_1L_1 \approx -X(0) - L_1^2, \quad (5.9)$$

and (ii) from the Einstein equations (4.7) yielding the result (5.3). Thus, the inflatory expansion rate H_1 is found as

$$H_1 \approx \left(\frac{2\pi}{3} \frac{L_p^2}{\Lambda_c a^3} \right)^{\frac{1}{2}} \approx \frac{3}{2} \frac{1}{\Lambda_c} \quad (5.10)$$

when all length parameters are (tentatively) put equal: $L_p = a = \Lambda_c$ ($\equiv \hbar/Mc$). Finally, the constant inflation value L_1 of the variable L is now found from (5.9) as

$$L_1 \approx \frac{1}{2\Lambda_c}, \quad (5.11)$$

where $X(0)$ has again been taken from (3.40a). With these arrangements the decay time for the short-living variables becomes $T_I \approx \Lambda_c/5$, and consequently their time behaviour is roughly given by $\exp[-\theta/T_I] = \exp[-5\theta/\Lambda_c]$. This means that the short-living variables experience a suppression by the factor $\exp[-50] \approx 10^{-20}$ during the inflatory phase of “duration” ten times the Compton length Λ_c ($\sim \theta \approx 12\Lambda_c$). At the same time the radius \mathcal{R} (5.2) of our model universe is increased by the factor $\exp[\frac{3}{2} \cdot 12] \approx 10^8$ (Fig. 1) and the “particle number” N_p has been increased from 1 to 10^{20} (see Figure 2).

Summarizing our results we see that our present model of the universe just represents a physical system demonstrating the phenomenon of Hamiltonian self-diagonalization: there are emerging two different time scales, namely the “Compton time” Λ_c and the duration of inflation ($\sim 12\Lambda_c$), such that there occurs a rapid de-coherence process ($\sim e^{-5\theta/\Lambda_c}$) into

the state of “independent constituents” which is described by the *diagonal* Hamiltonian

$$\mathcal{H}_\mu \rightarrow L b_\mu \cdot \mathbf{1}. \quad (5.12)$$

The diagonal part L changes not appreciably before the inflation comes to an end ($\theta \sim 12\Lambda_c$). It is easy to show that the present de-coherence effect is essentially due to the presence of inflation. In the subsequent standard phase ($\theta \gtrsim 12\Lambda_c$) the de-coherence is much weaker and for vanishing expansion rate ($H \rightarrow 0$) even totally absent.

5.4. Fierz deviation Δ_F^2

Besides the de-coherence effect there is another phenomenon which can occur in the inflation phase but not in the standard phase: this is the distinct change of the Fierz deviation Δ_F^2 (4.29). If this latter quantity is required to increase (or decrease) considerably in the course of time, one should have a definite sign for the Hamiltonian coefficient L . If L had permanently been negative the Fierz deviation Δ_F^2 would asymptotically tend to zero. This would imply that the densities could ultimately be generated by some wave function ψ . It seems that such a process is impossible in an *expanding* universe. The results of our numerical integrations of the equations of motion for the densities ρ (4.23) and s_j (4.26) say that the Fierz deviation remains *unchanged in the standard phase* (apart from those notorious oscillations) but its absolute value is increased exponentially during the inflation phase (Figure 6). Clearly, this outcome is readily understandable by means of (4.29) because, during the inflation phase, the Hamiltonian coefficient L remains (nearly) constant, and this is all that is necessary in order to drive Δ_F^2 off its initial value:

$$\Delta_F^2(\theta) \sim \exp[4L_1\theta] \sim \exp\left[2\frac{\theta}{\Lambda_c}\right] \quad (5.13)$$

(cf. (5.11)). In the subsequent standard phase the large (absolute) value of Δ_F^2 is then preserved forever.

6. Discussion

The preceding results clearly show the emergence of two rather distinct phases of our model universe which are of quite different character concerning the effects of *de-coherence* and *Fierz deviation*. The point

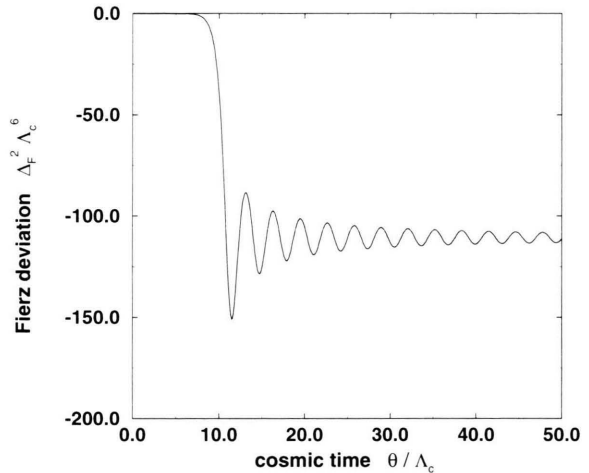


Fig. 6. The Fierz deviation Δ_F^2 . During the inflation phase ($\theta \lesssim 12\Lambda_c$) the absolute value of the Fierz deviation Δ_F^2 can be increased considerably (5.13) because the Hamiltonian coefficient L is of definite sign ($L = L_1 = \text{const}$). Whenever the time average of L vanishes the Fierz deviation must essentially remain unchanged (standard phase, $\theta \gtrsim 12\Lambda_c$).

with these effects is that, for the two-dimensional case, the complex (2×2) -matrix $\mathcal{H}_\mu = \mathcal{H} b_\mu$ (4.17) can be thought to be constituted by the 8 real objects $\{K, K_j, L, L_j, (j = 1, 2, 3)\}$ (4.18) which themselves split up into two subsets of rather distinguished time behaviour: the three variables $\{K, K_j, L_j\}$ are subject to a quite similar time evolution which stands in distinct contrast to the time evolution of the remaining variable L .

1. During the inflation phase the latter variable L adopts a constant value L_1 after a very short transient (Fig. 5). This constant value L_1 (5.11) together with the inflatory expansion rate H_1 (5.10) sets the time scale T_1 for the exponential decay of the other three Hamiltonian coefficients $K, K_j, L_j \sim \exp[-\theta/T_1]$. Thus, the Hamiltonian \mathcal{H} becomes rapidly proportional to the identity $\mathbf{1}$ ($\sim \mathcal{H} \Rightarrow \hbar c L \cdot \mathbf{1}$) which may be understood as a kind of dynamical self-diagonalization (“de-coherence”).
2. The inflation phase is followed by the standard phase which is characterized by an oscillatory time behaviour of the Hamiltonian variable L . As a consequence the striking physical effects (de-coherence and Fierz deviation) are not possible in the standard phase because their occurrence is

based upon a (nearly) constant value of L . Thus, the inflation phase is not only an exceptional state with respect to its geometry but also with respect to the accompanying physical processes.

These results could be obtained by extensive use of the technique of the Relativistic Schrödinger Equations which in that way suggest themselves as a useful tool for studying also *non-linear* field equations.

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